Quantum Mechanics and the Weak Equivalence Principle

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We use the Feynman path integral approach to nonrelativistic quantum mechanics twofold. First, we derive the lagrangian for a spinless particle moving in a uniformly but not necessarily constantly accelerated reference frame; then, applying the *strong equivalence principle* (SEP) we obtain the Schroedinger equation for a particle in an inertial frame and in the presence of a uniform and constant gravity field. Second, using the associated Feynman propagator, we propagate an initial gaussian wave packet, with the final wave function and probability density depending on the ratio $\frac{m}{\hbar}$, where m is the inertial mass of the particle, thus exhibiting the fact that *the weak equivalence principle* (WEP) *is violated by quantum mechanics*. Although due to rapid oscillations the wave function does not exist in the classical limit, the probability density is well defined and mass independent when $\hbar \to 0$, showing the recovery of the WEP. Finally, at the quantum level, a heavier particle does not necessarily falls faster than a lighter one; this depends on the relations between the initial and final common positions and times of the particles.

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I. INTRODUCTION

The *strong equivalence principle* (SEP) says that a reference frame accelerated with respect to an inertial system is equivalent to a uniform gravitational field in an inertial frame; and the other way around, an arbitrary gravitational field in an inertial system is locally equivalent to an accelerated frame. The *weak equivalence principle* (WEP), on the other hand, says that under identical initial conditions, the motion of particles in a given gravitational field is the same; in particular, it is independent of their inertial or gravitational masses[1]. In this context, it is important to emphasize the following points:
i) Both the SEP and the WEP are classical i.e. not quantum.
ii) The SEP implies the WEP: in fact, the accelerations of two particles with inertial masses m_1 and m_2 in a non inertial frame (and therefore in an equivalent gravitational field) are the same and independent of their masses; so their motions are equal.
iii) In the context of Newtonian mechanics it can be easily shown that the WEP is equivalent to the statement of the equality of the inertial mass m and the (passive) gravitational mass m_g[2] (see section 2).
iv) Even in the context of special relativity, it can be shown[3, 4] that in accelerated reference frames, the space (spacetime) geometry is not euclidean (not pseudoeuclidean) but riemannian (pseudoriemannian); this fact supports the geometrical Einstein's theory of gravitation i.e. general relativity[5].

As is well known, in nonrelativistic quantum mechanics (NRQM), the motion of a particle in the presence of an external gravitational field is mass dependent[6, 7]; this can be seen from the Schroedinger equation in the local gravitational potential $V = gx$. Once the SEP is accepted one obtains

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + mgx)\psi(x,t), \qquad (1)$$

i.e.

$$i\frac{\partial}{\partial t}\psi(x,t) = (-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{\hbar}gx)\psi(x,t), \qquad (2)$$

whose solution depends parametrically on $\frac{\hbar}{m}$. Moreover, even the free motion is mass dependent, not only in NRQM but also in relativistic quantum mechanics (RQM):

$$(\partial^2 + \lambda_c^{-2})\varphi(x^\mu) = 0, \quad Klein-Gordon \quad equation, \qquad (3)$$

and

$$(i\gamma\cdot\partial + \lambda_c^{-1})\psi(x^\mu) = 0, \quad Dirac \quad equation, \qquad (4)$$

where λ_c is the Compton length given by $\frac{\hbar}{mc}$. In NRQM the free propagator of a spinless particle of mass m is given by

$$K_0(x'',t'';x',t') = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}}\,e^{\frac{im}{\hbar}\frac{(x''-x')^2}{t''-t'}}, \qquad (5)$$

that replaces the mass independent solution of the corresponding free motion in classical mechanics:

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$$velocity = const. = \frac{x'' - x'}{t'' - t'}. \tag{6}$$

The above arguments show that in quantum mechanics, for wave functions, propagators and probability density distributions, *the WEP is violated*. However, this result does not imply neither the violation of the SEP, as claimed in Ref.[8], since the implication $SEP \Longrightarrow WEP$ is classical but not quantum, nor the violation of the WEP, at the level of expectation values, where it indeed holds[9]. Though the wave function and the propagator are not observables, but the position probability density distribution can be measured, then *the violation of the WEP in quantum mechanics is physical*. We do not discuss here the generalization of equivalence principles to quantum mechanics (see e.g. Ref.[10]).

In sec.2, assuming the SEP, we rederive the Eq.(2) in the context of the path integral formulation of quantum mechanics[11], which involves the classical lagrangian. We also show the equivalence between the equality $m = m_g$ (up to a universal constant) and the WEP.

In sec.3, we use the Feynman propagator in the presence of a uniform and constant gravitational field $\vec{g}$, and study the quantum free fall of a spinless particle of mass m that at time t' has a gaussian distribution of width σ around x' and average momentum p_0; the probability density at time $t'' > t'$ is also gaussian, but mass dependent. We also discuss the difference in the final probability density distributions ρ_1 and ρ_2 corresponding to two different particles of masses m_1 and m_2, with $m_1 > m_2$. A similar analysis but in the framework of the causal interpretation of quantum mechanics (Bohmian mechanics) was done in Ref.[12].

Finally, in sec.4 the classical limit $\hbar \to 0$ is taken and the mass dependence of quantum free fall which is discussed in sec. 3 disappears, as it should be to recover the classical WEP. The limit $\sigma \to 0$ for the ideal case of perfectly localized particles is shown to behave in the same way.

II. UNIFORM ACCELERATION AND SEP IN NON-RELATIVISTIC QUANTUM MECHANICS

Let $t'' > t'$; then, in an inertial reference frame with coordinates $(\vec{x}, t)$ the Feynman propagator between the points $(\vec{x}', t')$ and $(\vec{x}'', t'')$ of a non-relativistic particle with inertial mass m in a time independent potential U can be represented by an integral over all continuous paths joining the above initial and final points[11]:

$$K(\vec{x}'', t''; \vec{x}', t') = \int_{\vec{y}(t')=\vec{x}'}^{\vec{y}(t'')=\vec{x}''} \mathcal{D}\vec{y}(t)\; exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m|\dot{\vec{y}}(t)|^2 - U(\vec{y} - \vec{y}_0)\right)\right]. \tag{7}$$

$\vec{y}_0$ is an arbitrary reference point. Going to a reference frame with spacetime coordinates

$$\tilde{\vec{y}} = \vec{y} - \vec{\xi}(t) \tag{8}$$

and

$$\tilde{t} = t \tag{9}$$

where $\vec{\xi}(t)$ is an arbitrary twice differentiable function of the time, after integration, Eq.(7) becomes

$$K(\vec{x}'', t''; \vec{x}', t') = exp\left[\frac{im}{\hbar}\left(\tilde{\vec{x}}'' \cdot \dot{\vec{\xi}}(t'') + \frac{1}{2}\int_{t'}^{t''} dt |\dot{\vec{\xi}}(t)|^2\right)\right]$$
$$\times \int_{\tilde{\vec{y}}(t')=\tilde{\vec{x}}'}^{\tilde{\vec{y}}(t'')=\tilde{\vec{x}}''} \mathcal{D}\tilde{\vec{y}}\; exp\left[\frac{i}{\hbar}\int_{t'}^{t''} dt\left(\frac{m}{2}|\dot{\tilde{\vec{y}}}|^2 - (m\tilde{\vec{y}} \cdot \ddot{\vec{\xi}} + U(\tilde{\vec{y}} - \tilde{\vec{y}}_0))\right)\right] e^{-\frac{im}{\hbar}\tilde{\vec{x}}' \cdot \dot{\vec{\xi}}(t')}, \tag{10}$$

where $\tilde{\vec{x}}' = \vec{x}' - \vec{\xi}(t')$, $\tilde{\vec{x}}'' = \vec{x}'' - \vec{\xi}(t'')$ and $\mathcal{D}\vec{y} = \mathcal{D}\tilde{\vec{y}}$. Defining the wave function

$$\tilde{\psi}(\tilde{\vec{x}}, \tilde{t}) = e^{-\frac{im}{\hbar}(\tilde{\vec{x}} \cdot \dot{\vec{\xi}}(t) + \frac{1}{2}\int_{\bar{t}}^{\tilde{t}} d\tau |\dot{\vec{\xi}}(\tau)|^2)}\psi(\vec{x}, t) \tag{11}$$

where $\bar{t}$ is an arbitrary instant between t' and t'', one obtains

$$\tilde{\psi}(\tilde{\vec{x}}'', \tilde{t}'') = \int d\tilde{\vec{x}}' \tilde{K}(\tilde{\vec{x}}'', \tilde{t}''; \tilde{\vec{x}}', t')\tilde{\psi}(\tilde{\vec{x}}', \tilde{t}') \tag{12}$$

with

$$\tilde{K}(\vec{\tilde{x}}'',\tilde{t}'';\vec{\tilde{x}}',\tilde{t}') = \int_{\vec{\tilde{x}}'}^{\vec{\tilde{x}}''} \mathcal{D}\vec{\tilde{y}} e^{\frac{i}{\hbar}\int_{\tilde{t}'}^{\tilde{t}''} dt (\frac{m}{2}|\dot{\vec{\tilde{y}}}(t)|^2 - U_{eff})} \quad (13)$$

and where U_{eff} is an effective potential naturally incorporating the effect of the acceleration $\ddot{\vec{\xi}}(t)$:

$$U_{eff} = m\vec{\tilde{y}}\cdot\ddot{\vec{\xi}} + U(\vec{\tilde{y}} - \vec{\tilde{y}}_0). \quad (14)$$

The SEP allows us to reinterpret $\tilde{K}$ as the Feynman propagator in an inertial frame in the presence of a uniform but otherwise arbitrary gravitational field given by $\ddot{\vec{\xi}}(t)$[1]. In particular, for constant

$$\ddot{\vec{\xi}} = \vec{g} \quad (15)$$

we obtain the usual Feynman propagator for a spinless quantum particle coupled to a constant and uniform gravitational field $\vec{g}$:

$$K(\vec{x}'',t'';\vec{x}',t';\vec{g};\frac{m}{\hbar}) = \int_{\vec{y}(t')=\vec{x}'}^{\vec{y}(t'')=\vec{x}''} \mathcal{D}\vec{y}\ exp\left[(\frac{i}{\hbar}\int_{t'}^{t''} dt\left(\frac{m}{2}|\dot{\vec{y}}(t)|^2 - (m\vec{y}(t)\cdot\vec{g} + U(\vec{y}-\vec{y}_0))\right)\right], \quad (16)$$

where we emphasized the dependence of K on $\vec{g}$ and on the ratio $\frac{m}{\hbar}$.

Notice that the assumption of the validity of the SEP also in quantum mechanics, has implied the equality of the inertial mass m with the *passive gravitational mass* m_g, which gives the coupling between the particle and the gravitational field:

$$U_g = m_g\vec{\tilde{y}}\cdot\vec{g}. \quad (17)$$

Classically, the equality $m = m_{gr}$ (or $m = km_{gr}$ with k a universal constant) is *equivalent* to the WEP, which says that under identical initial conditions the motion (acceleration) of particles in a given gravitational field is independent of their masses[2]. In fact, for two particles with inertial masses m and M and corresponding passive gravitational masses m_g and M_g, the Newton equations are $a = \frac{m_g}{m}g$ and $A = \frac{M_g}{M}g$; then the WEP implies $a = A = \tilde{g}$ and therefore $\nu_g = k\nu$ for both $\nu = m$ and $\nu = M$ ($\tilde{g} = kg$ and $\tilde{g} = g$ only if the units are chosen such that $k = 1$); the other way around: if $\nu_g = k\nu$ then $\nu a = \nu_g g = k\nu g$ and then $a = kg$ for both m and M. So, classically,

$$(m_g = km) \overset{cl}{\Longleftrightarrow} WEP. \quad (18)$$

But also we have that both in quantum mechanics as well as classically,

$$SEP \overset{QM/cl}{\Longrightarrow} (m_g = km). \quad (19)$$

Then we have the chain of implications

$$SEP \overset{QM/cl}{\Longrightarrow} (m_g = km) \overset{cl}{\Longleftrightarrow} WEP. \quad (20)$$

The Schroedinger equation which emerges from Eq.(16) is[11]

$$i\frac{\partial}{\partial t}\psi(\vec{y},t) = (-\frac{1}{2}\frac{\hbar}{m}\nabla^2 + (\frac{m}{\hbar}),\vec{y}\cdot\vec{g})\psi(\vec{y},t) \quad (21)$$

where we have set $U(\vec{y}-\vec{y}_0) = 0$. This formula has been experimentally verified by the now famous COW experiment[7] using neutron interferometry. Both the Schroedinger equation as well as the propagator $\tilde{K}$ are mass dependent which strongly suggests that the WEP is violated by quantum mechanics. But this does not invalidate neither the equality $m_g = km$ (since the implication $(m_g = km) \overset{cl}{\Longrightarrow} WEP$ involves the classical Newton equation) nor the SEP which remains true both in classical mechanics as well as in quantum mechanics. Obviously, in QM we have the result that

$$(m_g = km) \ \ is\ \ not\ \ equivalent\ \ to\ \ WEP. \quad (22)$$

This is discussed in the book by Sakurai[6].

III. PROPAGATION OF A GAUSSIAN WAVE PACKET IN THE LOCAL GRAVITATIONAL FIELD AND VIOLATION OF THE WEP

To study the free fall of a quantum particle of mass m in a local gravitational potential gx, we need the propagator $K(x'',t'';x',t')$. For simplicity, we shall consider the whole vertical axis x as the domain of the motion, ignoring the infinite barrier imposed by the surface of the earth.

For a quadratic lagrangian of the form

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}(t)^2+b(t)x(t)\dot{x}(t)+d(t)\dot{x}(t)-\frac{1}{2}c(t)x(t)^2-e(t)x(t)-f(t) \tag{23}$$

the propagator is given by[11, 13]

$$K(x'',t'';x',t')=\sqrt{\frac{m}{2\pi i\hbar f(t'',t')}}e^{\frac{i}{\hbar}S[\bar{x}]} \tag{24}$$

where $\bar{x}(t)$ is the classical path joining the initial and final points (x',t') and (x'',t''), and $f(\xi,\eta)$ satisfies the differential equation

$$\frac{\partial^2}{\partial\xi^2}f(\xi,\eta)+\frac{\dot{b}(\xi)+c(\xi)}{m}f(\xi,\eta)=0, \tag{25}$$

with the conditions

$$f(\xi,\xi)=0, \quad \frac{\partial}{\partial\xi}f(\xi,\eta)|_{\xi=\eta}=1. \tag{26}$$

In our case, from Eq.(16) with $U(\vec{y}-\vec{y}_0)=0$, the lagrangian becomes

$$L=\frac{m}{2}\dot{x}^2-mgx(t), \tag{27}$$

i.e. $b(t)=d(t)=c(t)=f(t)=0$ and $e(t)=mg$; then the Eq.(25) reduces to $\frac{\partial^2}{\partial\xi^2}f(\xi,\eta)=0$ with solution $f(\xi,\eta)=\xi\alpha(\eta)+\beta(\eta)$; from the initial conditions, $\eta\alpha(\eta)+\beta(\eta)=0$ which implies $\beta(\eta)=-\eta\alpha(\eta)$ and $\alpha(\eta)=1$, then $\beta(\eta)=-\eta$ and so $f(\xi,\eta)=\xi-\eta$. Then $f(t'',t')=t''-t'$ and the propagator becomes

$$K(x'',t'';x',t')=\sqrt{\frac{m}{2\pi i\hbar(t''-t')}}e^{\frac{i}{\hbar}S[\bar{x}]} \tag{28}$$

with

$$\bar{x}(t)=x'+v_0(t-t')-\frac{1}{2}g(t-t')^2, \tag{29}$$

where v_0 is the initial velocity given by

$$v_0=\frac{x''-x'}{t''-t'}+\frac{g}{2}(t''-t'). \tag{30}$$

For the classical action one has

$$S[\bar{x}]=\int_{t'}^{t''}dt\left[\frac{1}{2}\dot{\bar{x}}(t)^2-mg\bar{x}(t)\right]=\frac{m}{2}(t''-t')\left[\left(\frac{x''-x'}{t''-t'}\right)^2-g(x''+x')-\frac{1}{12}g^2(t''-t')^2\right] \tag{31}$$

which gives the propagator, explicitly depending on $\frac{m}{\hbar}$,

$$K(x'',t''-t',x';\frac{m}{\hbar})=\sqrt{\frac{m}{2\pi i\hbar(t''-t')}}e^{\frac{im}{2\hbar}(t''-t')\left[\left(\frac{x''-x'}{t''-t'}\right)^2-g(x''+x')-\frac{1}{12}g^2(t''-t')^2\right]}. \tag{32}$$

In the classical limit, K does not exist by the rapid oscillations of the exponential; however,

$$|K(x'',t''-t',x';\frac{m}{\hbar})| \overset{\hbar\to 0}{\longrightarrow} +\infty. \tag{33}$$

We remark that even the free nonrelativistic propagator depends on m:

$$K_0(x''-x',t''-t';\frac{m}{\hbar})=\sqrt{\frac{m}{2\pi i\hbar(t''-t')}}e^{\frac{im}{2\hbar}\frac{(x''-x')^2}{t''-t'}} \tag{34}$$

which says that even the free motion is mass dependent. The same happens for the propagators in the relativistic domain, like those for the Klein-Gordon and Dirac particles. This is a first indication of the violation of the WEP in the quantum regime.

If $\psi(x',t')$ is the initial wave function describing our "falling" particle, then the wave function at (x'',t'') is given by

$$\psi(x'',t'')=\int_{-\infty}^{+\infty}dx'K(x'',t'';x',t')\psi(x',t'). \tag{35}$$

For $\psi(x',t')$ we choose a normalized gaussian wave packet centered at x', average momentum $p_0=\hbar k_0$ and therefore average velocity $u_0=\frac{p_0}{m}=\frac{\hbar k_0}{m}$, and width σ, namely

$$\psi(y,t') = \frac{e^{-\frac{(y-x')^2}{2\sigma^2}+ik_0 y}}{\pi^{\frac{1}{4}}\sqrt{\sigma}}. \tag{36}$$

Then, the wave function at (x'',t'') is

$$\psi(x'',t''-t',x';g;\sigma;\frac{m}{\hbar};k_0) = \int_{-\infty}^{+\infty} dy K(x'',t'';y,t')\psi(y,t')$$
$$= \sqrt{\frac{m}{2\pi^{\frac{3}{2}}\hbar i\sigma(t''-t')}} e^{\frac{im}{2\hbar}\left[\frac{x''}{(t''-t')}\left(x''-g(t''-t')^2\right)-\frac{g}{12}(t''-t')^3\right]-\frac{(x')^2}{2\sigma^2}} \int_{-\infty}^{+\infty} dy e^{Ay^2+By} \tag{37}$$

where

$$A = -\frac{1}{2\sigma^2}+\frac{im}{2\pi(t''-t')} \quad and \quad B = \frac{x'}{\sigma^2}+i(\frac{-m}{2\hbar(t''-t')}(2x''+g(t''-t')^2)+k_0). \tag{38}$$

Using the analytic continuation of the integral

$$\int_{-\infty}^{+\infty} dy e^{-ay^2+iby} = \sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}, \quad a>0, \quad b\in\mathbb{R} \tag{39}$$

to the domain $a\in\mathbb{C}$, $Re(a)>0$, $b\in\mathbb{C}$, we obtain

$$\psi(x'',\delta t,x';g;\sigma;\frac{m}{\hbar};k_0) = \sqrt{\frac{\sigma}{i\sqrt{\pi}\delta t\frac{\hbar}{m}+\sqrt{\pi}\sigma^2}}e^{\mathcal{F}}e^{\mathcal{G}}, \tag{40}$$

where for convenience we have defined

$$\delta t = t''-t', \tag{41}$$

$$\begin{aligned}\mathcal{F} \;=\; &-\frac{1}{2}\Bigg[\frac{x'^2}{\sigma^2}+\Bigg(\Bigg\{-x'^2(\delta t)^2+\frac{m^2}{4\hbar^2}\sigma^4(2x''+g(\delta t)^2)^2\\ &+k_0\sigma^4\delta t[k_0\delta t-\frac{m}{\hbar}(2x''+g(\delta t)^2)]\Bigg\}\sigma^2(\delta t)^2\\ &-x'\frac{m^2}{\hbar^2}\sigma^6(\delta t)^2(2x''+g(\delta t)^2)+\frac{2m}{\hbar}\sigma^6 k_0 x'(\delta t)^3\Bigg)[\sigma^4(\delta t)^4(1+\frac{m^2\sigma^4}{\hbar^2(\delta t)^2})]^{-1}\Bigg],\end{aligned} \tag{42}$$

and

$$\begin{aligned}\mathcal{G} \;=\; &-\frac{i}{2}\Bigg[x'(\delta t)^3\frac{m}{\hbar}\sigma^4(2x''+g(\delta t)^2)-\frac{m}{\hbar}\sigma^4 x'^2(\delta t)^3+\frac{m^3}{4\hbar^3}\sigma^8\delta t(2x''+g(\delta t)^2)^2\\ &+\left(k_0^2\sigma^4(\delta t)^2-\frac{m}{\hbar}\sigma^4 k_0\delta t(2x''+g(\delta t)^2)\right)\frac{m}{\hbar}\sigma^4\delta t\\ &-2\sigma^4 k_0 x'(\delta t)^4\Bigg][\sigma^4(\delta t)^4(1+\frac{m^2\sigma^4}{\hbar^2(\delta t)^2})]^{-1}.\end{aligned} \tag{43}$$

The factor before the exponentials has a well defined classical limit since

$$\sqrt{\frac{\sigma}{i\sqrt{\pi}(t''-t')\frac{\hbar}{m}+\sqrt{\pi}\sigma^2}} \xrightarrow{\hbar\to 0} \frac{\pi^{-\frac{1}{4}}}{\sqrt{\sigma}}; \tag{44}$$

however, the imaginary exponential does not exist in this limit since it behaves as

$$exp\left[-\frac{i}{2}\frac{m}{\hbar}\frac{\left(x''+\frac{g}{2}(t''-t')^2\right)^2}{t''-t'}+i\frac{p_0}{\hbar}\left(x''+\frac{g}{2}(t''-t')^2\right)\right] \tag{45}$$

which oscillates indefinitely in the unit circle when $\hbar \to 0$ (unless $u_0 = \frac{1}{2}\frac{x''+\frac{g}{2}(t''-t')^2}{t''-t'}$, in which case is equal to 1).

This means that the wave function does not exist in this limit. However, this is not so for the classical limit of the probability density. After some reordering of the terms in the real exponential, the square of the absolute value of the wave function, namely, the probability density (ρ) is given by

$$\rho(x'', t''-t', x'; g; \sigma; \mu; u_0) = |\psi(x'', t''-t', x'; g; \sigma, \mu; u_0)|^2$$
$$= \frac{1}{\sqrt{\pi}\sqrt{\frac{(t''-t')^2}{\mu^2\sigma^2}+\sigma^2}}\; exp\left[-\frac{\left(x''-\left[x'+u_0(t''-t')-\frac{g}{2}(t''-t')^2\right]\right)^2}{\frac{(t''-t')^2}{\mu^2\sigma^2}+\sigma^2}\right], \tag{46}$$

where we have defined $\mu = m/\hbar$. Clearly,

$$\int_{-\infty}^{+\infty} dx''\rho(x'', t''-t', x'; g; \sigma; \mu; u_0) = 1, \tag{47}$$

i.e. the normalization of the initial wave function is preserved. Also,

$$\rho(x'', t''-t', x'; g; \sigma; \mu; u_0) = \rho(x''-x', t''-t'; g; \sigma; \mu; u_0). \tag{48}$$

Eq.(46) illustrates the *violation of the WEP*: the probability density to find the "falling" quantum particle at x'' and at time t'' depends on the mass m through the ratio μ, showing that quantum non-relativistic particles "fall" differently for different values of the mass. In the next section we show, however, that the WEP is recovered in the limit $\hbar \to 0$. The width Σ of the probability density ρ decreases with the mass and is given by

$$\Sigma = \sigma\sqrt{1+\frac{\hbar^2(t''-t')^2}{m^2\sigma^4}}. \tag{49}$$

Notice that Σ is independent of g and coincides with the broadening of a free gaussian wave packet. Since ρ is normalized, for $m_1 > m_2$, ρ_1 is more peaked than ρ_2 and so $\rho_1(x'') > \rho_2(x'')$ for $|x''-\bar{x''}| < \Delta$ and $\rho_1(x'') < \rho_2(x'')$ for $|x''-\bar{x''}| > \Delta$ with

$$\Delta = \frac{\Sigma_1\Sigma_2}{\sqrt{\Sigma_2^2-\Sigma_1^2}}\sqrt{ln(\frac{\Sigma_2}{\Sigma_1})}, \tag{50}$$

defined by $\rho_1(\bar{x''}\pm\Delta) = \rho_2(\bar{x''}\pm\Delta)$. Here,

$$\bar{x''} = x' + u_0(t''-t') - \frac{g}{2}(t''-t')^2. \tag{51}$$

Far away from the center of the distributions, the lighter particle has more probability to be found i.e. to have fallen, than the heavier one.

In terms of $\bar{x''}$ and Σ, the probability density at x'' is given by

$$\rho(x''; \bar{x''}, \Sigma) = \frac{1}{\sqrt{\pi}\Sigma}e^{-\frac{(x''-\bar{x''})^2}{\Sigma^2}}. \tag{52}$$

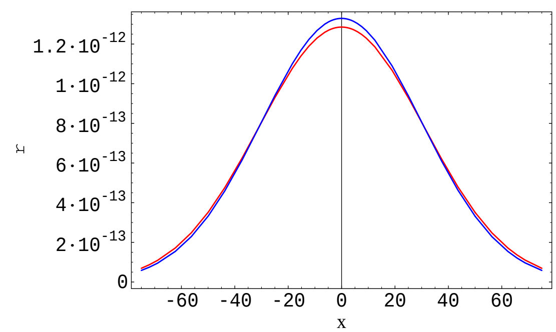

FIG. 1: The probability density $\rho \equiv r$ is plotted as a function of $x'' \equiv x$ for $\pi^{\pm}$ (upper curve at x=0) and π^0.

In particular, when the average velocity u_0 of the initial wave packet equals the initial velocity v_0 of the classical solution, $\bar{x''} = x''$ and ρ reaches its maximum value:

$$\rho(x''; x'', \Sigma) = \rho_{max} = \frac{1}{\sqrt{\pi}\Sigma} \quad for \quad u_0 = v_0. \tag{53}$$

For a qualitative picture of the above mentioned behavior of quantum falling, in Fig. 1, we plot ρ as a function of x'' (in units of meters) for π^0 (mass 134.98 MeV/c^2) and $\pi^{\pm}$ (mass 139.57 MeV/c^2); in Fig. 2 , we plot ρ for π^0 and K^0 (mass 497.67 MeV/c^2). In both cases we have chosen $x' = 0$ and u_0 such that $\bar{x''} = 0$. We have also taken $\sigma = 10^2$ Å. In both figures, it is clear that far away from the center of the distributions, the lighter particle has higher probability to be found than the heavier one.

IV. CLASSICAL LIMIT AND RECOVERY OF THE WEP

The Eq.(46) for the probability density at (x'', t'') for the freely falling quantum particle in the constant and

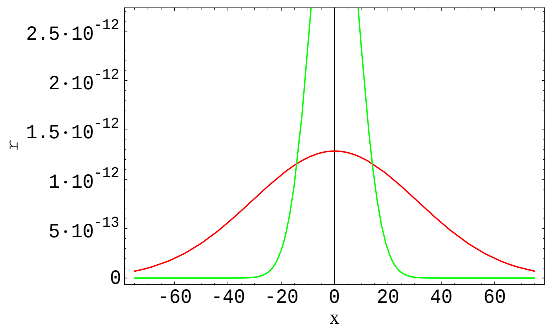

FIG. 2: The probability density $\rho \equiv r$ is plotted as a function of $x'' \equiv x$ for π^0 and K^0 (upper curve at x=0).

uniform gravitational field $\vec{g} = -g\hat{x}$, has a well defined *mass independent* classical limit given by

$$lim_{\hbar\to 0}\rho(x''-x',t''-t';g;\sigma;\frac{m}{\hbar};u_0) = \frac{1}{\sqrt{\pi}\sigma}exp - \frac{(x''-\bar{x''})^2}{\sigma^2} \equiv \rho_{cl}(x''-x',t''-t';g;\sigma;u_0). \tag{54}$$

The gaussian in Eq.(54) has the same width as the initial probability distribution, but is centered around $\bar{x''}$. The absence of the mass in ρ_{cl} exhibits the recovery of the WEP in the classical limit[14]. For the case of an initial perfectly localized particle, with probability density

$$lim_{\sigma\to 0}|\psi(y,t')|^2 = \frac{1}{\sqrt{\pi}}lim_{\sigma\to 0}\frac{e^{\frac{(y-x')^2}{\sigma^2}}}{\sigma} = \delta(y-x'), \tag{55}$$

one also obtains a perfectly localized particle in the classical limit:

$$\begin{aligned} lim_{\sigma\to 0}\rho_{cl}(x''-x',t''-t';g;\sigma;u_0) &= \frac{1}{\sqrt{\pi}}lim_{\sigma\to 0}\frac{e^{\frac{(x''-\bar{x''})^2}{\sigma^2}}}{\sigma} \\ &= \delta(x''-\bar{x''}) = \delta\left(x''-\left[x'+u_0(t''-t')-\frac{g}{2}(t''-t')^2\right]\right). \end{aligned} \tag{56}$$

So our probability density in Eq.(56) is independent of mass and localized in space, which is consistent with the classical description of the particle.

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